

Rotating cylindrical wormholes: a no-go theorem

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The existing solutions to the Einstein equations describing rotating cylindrical wormholes are not asymptotically flat and therefore cannot describe wormhole entrances as local objects in our Universe. To overcome this difficulty, flat asymptotic regions are added to wormhole solutions by matching them at some surfaces Σ_- and Σ_+ . It is shown, however, that if the wormhole solution is obtained for scalar fields with arbitrary potentials, possibly interacting with an azimuthal electric or magnetic field, then the matter content of one or both thin shells appearing on Σ_- and Σ_+ violate the Null Energy Condition. Thus exotic matter is still necessary for obtaining a twice asymptotically flat wormhole.

Traversable Lorentzian wormholes are widely discussed in gravitational physics since they lead to many effects of interest like time machines or shortcuts between distant parts of space. Large enough wormholes, if any, can lead to observable effects in astronomy [1–3].

In attempts to build realistic wormhole models, the main difficulty is that in general relativity (GR) and some of its extensions a static wormhole geometry requires the presence of “exotic”, or phantom matter, that is, matter violating the weak and null energy condition (WEC and NEC), at least near the throat, the narrowest place in a wormhole [4–7]. This result is obtained if the throat is a compact 2D surface with a finite area [6].

Examples of phantom-free wormhole solutions have been obtained in extensions of GR, such as the Einstein-Cartan theory [8], Einstein-Gauss-Bonnet gravity [9], brane worlds [10], etc. We here prefer to consider GR as a theory quite well describing the reality on the macroscopic scale while the extensions more likely concern very large densities and/or curvatures. In GR there are phantom-free wormhole models with axial symmetry, including the Zipoy [11] and superextremal Kerr vacuum solutions as well as solutions with scalar and electromagnetic fields possibly interacting with each other [12,13]; in all of them, however, a disk that plays the role of a wormhole throat is bounded by a ring singularity whose existence may be thought of as an unpleasant price paid for the absence of exotic matter.

The above-mentioned result of [6] does not directly apply to objects like cosmic strings, infinitely stretched along a certain direction, in the simplest case cylindrically symmetric ones. Such systems with and without rotation were discussed, in particular, in [14–17] (see also references therein). It was shown there, with a number of examples, that phantom-free cylindrical wormhole solutions to the Einstein equations are rather easily obtained.

A problem with cylindrical systems is their undesirable asymptotic behavior. To describe wormholes potentially visible to distant observers like ourselves in our very weakly curved Universe, one has to require their flat (or string²) asymptotics, which is hard to achieve under cylindrical symmetry.

To overcome this difficulty, it was suggested [15] to build wormholes with flat asymptotic regions on both sides of the throat by matching a wormhole solution to suitable parts of Minkowski space-time, but no successful examples of phantom-free wormholes were so far obtained in this way. In the present paper we further discuss this opportunity and prove a no-go theorem on the conditions under which this cut-and-paste procedure cannot lead to asymptotically flat phantom-free wormholes.

Consider a stationary cylindrically symmetric metric of the form

$$ds^2 = e^{2\gamma(u)}[dt - E(u)e^{-2\gamma(u)}d\varphi]^2 - e^{2\alpha(u)}du^2 - e^{2\mu(u)}dz^2 - e^{2\beta(u)}d\varphi^2, \quad (1)$$

where u , z and φ are the radial, longitudinal and angular coordinates. This metric is said to describe a wormhole if either (i) the circular radius $r(u) = e^{\beta(u)}$ has a minimum (called an r -throat) and is large

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²Or at least asymptotically flat up to an angular deficit well-known in cosmic string configurations. In what follows, for brevity, we speak of asymptotic flatness though cosmic-string asymptotics could also be mentioned on equal grounds.

(preferably tends to infinity) far from this minimum or (ii) a similar behavior is observed for the area function $a(u) = e^{\mu+\beta}$ (its minimum is called an a -throat) [14,15]. If a wormhole is asymptotically flat at both extremes of the u range, it evidently possesses both kinds of throats.

The vortex gravitational field existing in a space-time with the metric (1) can be characterized by the angular velocity $\omega(u)$ given by [15,18,19]

$$\omega = \frac{1}{2}(Ee^{-2\gamma})'e^{\gamma-\beta-\alpha}. \quad (2)$$

under an arbitrary choice of the coordinate u (a prime stands for d/du). Furthermore, the reference frame comoving to a matter distribution in its motion by the angle φ is determined by a zero component T_0^3 of the stress-energy tensor (SET), hence (via the Einstein equations) by the Ricci tensor component $R_0^3 \sim (\omega e^{2\gamma+\mu})' = 0$, and thus in this reference frame we have

$$\omega = \omega_0 e^{-\mu-2\gamma}, \quad \omega_0 = \text{const}. \quad (3)$$

It then turns out [15] that the diagonal components of the Ricci (R_μ^ν) and Einstein ($G_\mu^\nu = R_\mu^\nu - \frac{1}{2}\delta_\mu^\nu R$) tensors split into the corresponding components for the static metric (that is, (1) with $E = 0$) plus the ω -dependent addition, so that $G_\mu^\nu = {}_sG_\mu^\nu + \omega G_\mu^\nu$, where ${}_sG_\mu^\nu$ is the static part, and

$$\omega G_\mu^\nu = \omega^2 \text{diag}(-3, 1, -1, 1). \quad (4)$$

Moreover, the tensors ${}_sG_\mu^\nu$ and ωG_μ^ν (each separately) satisfy the conservation law $\nabla_\alpha G_\mu^\alpha = 0$ in this static metric. Hence, by the Einstein equations $G_\mu^\nu = -\varkappa T_\mu^\nu$, the tensor $\omega G_\mu^\nu/\varkappa$ acts as an additional SET with exotic properties (e.g., the effective energy density is $-3\omega^2/\varkappa < 0$), making it easier to satisfy the existence conditions for both r - and a -throats, as confirmed by a number of examples in [15,17,19].

In all these examples, however, the wormholes are not asymptotically flat; even more than that, as is clear from Eq. (3), any solution obtained in this way cannot be asymptotically flat since it would require $\omega \rightarrow 0$ along with finite limits of γ and μ , which is incompatible with (3).

The following trick was suggested [15] for obtaining an asymptotically flat configuration: to cut a non-asymptotically flat wormhole configuration at some regular cylinders Σ_+ ($u = u_+$) and Σ_- ($u = u_-$) on different sides of the throat and to match it there with flat-space regions extending to infinity. Then the junction surfaces comprise thin shells with certain surface SETs, and it remains to check whether or not these SETs satisfy the WEC and NEC.

We will show, however, that with a large class of matter sources of the wormhole solution, it is impossible to obtain the surface SETs on both Σ_+ and Σ_- respecting the NEC.

Indeed, if we choose the harmonic radial coordinate u [20,21] specified by the relation $\alpha = \beta + \gamma + \mu$, then a certain combination of the Einstein equations takes the form

$$\beta'' - \gamma'' - 4\omega_0^2 e^{2\beta-2\gamma} = \varkappa e^{2\alpha} (T_t^t - T_\varphi^\varphi). \quad (5)$$

Therefore, if the SET of matter satisfies the condition $T_t^t = T_\varphi^\varphi$, Eq. (5) is easily integrated giving

$$e^{\beta-\gamma} = \frac{1}{2|\omega_0|s(k,u)}, \quad k = \text{const}, \quad (6)$$

where one more integration constants has been excluded by choosing the origin of the u coordinate, and the function $s(k,u)$ is defined as

$$s(k,u) = \begin{cases} k^{-1} \sinh ku, & k > 0, \quad u \in \mathbb{R}_+; \\ u, & k = 0, \quad u \in \mathbb{R}_+; \\ k^{-1} \sin ku, & k < 0, \quad 0 < u < \pi/|k|. \end{cases} \quad (7)$$

The condition $T_t^t = T_\varphi^\varphi$ holds for a large class of matter Lagrangians in the metric (1), such as, e.g.,

$$L = g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - 2V(\phi) - P(\phi) F^{\mu\nu} F_{\mu\nu}, \quad (8)$$

with an arbitrary scalar field potential $V(\phi)$ and an arbitrary function $P(\phi)$ characterizing the scalar-electromagnetic interaction, assuming that $\phi = \phi(u)$ and that the Maxwell tensor $F_{\mu\nu}$ describes a stationary azimuthal magnetic field ($F_{21} = -F_{12} \neq 0$) or its electric analogue.

Quite a general observation follows from Eq. (6): in the case $k < 0$, if the remaining field equations give the metric functions μ and α which are finite and regular on the segment $0 \leq u \leq \pi/|k|$ (or, equivalently, on any other half-wave of the function $\sin ku$), then the whole configuration is a wormhole, with both r - and a -throats; both $r(u)$ and $a(u)$ tend to infinity as $u \rightarrow 0$ and $u \rightarrow \pi/|k|$, but these limiting surfaces are singular due to $e^\gamma \rightarrow 0$ and $\omega \rightarrow \infty$. Configurations with $k \geq 0$ can also be of wormhole nature, as is confirmed by examples of vacuum and scalar-vacuum solutions [15, 17].

Now suppose we have such a wormhole metric and cut it at some regular points $u = u_-$ to the left and $u = u_+$ to the right of both r - and a -throats. Let us try to join it at $u = u_\pm$ to regions of Minkowski space-time with the metric $ds_M^2 = dt^2 - dx^2 - dz^2 - x^2 d\varphi^2$ outside suitable cylinders $x = x_\pm = \text{const}$. Assuming a rotating reference frame with an angular velocity $\Omega = \text{const}$, we substitute $\varphi \rightarrow \varphi + \Omega t$ to obtain

$$ds_M^2 = dx^2 + dz^2 + x^2(d\varphi + \Omega dt)^2 - dt^2. \quad (9)$$

The relevant quantities in the notations of (1) are

$$e^{2\gamma} = 1 - \Omega^2 x^2, \quad e^{2\beta} = \frac{x^2}{1 - \Omega^2 x^2}, \quad E = \Omega x^2, \quad \omega = \frac{\Omega}{1 - \Omega^2 x^2}. \quad (10)$$

This metric is stationary and can be matched to an internal metric at $|x| < 1/|\Omega|$, inside the “light cylinder” on which the linear rotational velocity reaches that of light.

Matching of two cylindrically symmetric regions at a surface $\Sigma : u = u_0$ means that we identify the two metrics on this surface, so that

$$[\beta] = 0, \quad [\mu] = 0, \quad [\gamma] = 0, \quad [E] = 0, \quad (11)$$

where, as usual, the brackets denote discontinuities: for any $f(u)$, $[f] = f(u_0 + 0) - f(u_0 - 0)$. One should note that in general the metrics on different sides of the surface Σ may be written using different choices of the radial coordinate u , but it does not matter since the quantities involved in all matching conditions used are insensitive to the choice of u .

The next step is to determine the material content of the junction surface Σ according to the Darmois-Israel formalism [22, 23]: in our case of a timelike surface Σ , the surface SET S_a^b is expressed in terms of K_a^b , the extrinsic curvature of Σ , as

$$S_a^b = -\kappa^{-1}[\tilde{K}_a^b], \quad \tilde{K}_a^b := K_a^b - \delta_a^b K_c^c, \quad a, b, c = 0, 2, 3. \quad (12)$$

Assume that the matching conditions (11) on Σ_\pm , identifying the surfaces $x = x_\pm$ in Minkowski regions and $u = u_\pm$ in the internal region, are fulfilled by fixing the values of x_\pm for given u_\pm and other parameters of the system, including the values of $\Omega = \Omega_\pm$ in each Minkowski region. It is important that we should take $x_+ > 0$ and $x_- < 0$ to adjust the directions of the normal vectors to Σ_\pm .

Now, the question is whether the surface SETs on Σ_\pm can satisfy the WEC whose requirements are

$$S_{00}/g_{00} = \sigma \geq 0, \quad S_{ab}\xi^a\xi^b \geq 0, \quad (13)$$

where ξ^a is any null vector ($\xi^a\xi_a = 0$) on $\Sigma = \Sigma_\pm$; the second inequality in (13) comprises the NEC as part of the WEC. The conditions (13) are equivalent to

$$[\tilde{K}_{44}/g_{44}] \leq 0, \quad [K_{ab}\xi^a\xi^b] \leq 0. \quad (14)$$

If we choose two null vectors on Σ in the z and φ directions as

$$\xi_{(1)}^a = (e^{-\gamma}, e^{-\mu}, 0), \quad \xi_{(2)}^a = (e^{-\gamma} + Ee^{-\beta-2\gamma}, 0, e^{-\beta}), \quad (15)$$

the conditions (14) read [15]

$$[e^{-\alpha}(\beta' + \mu')] \leq 0, \quad [e^{-\alpha}(\mu' - \gamma')] \leq 0, \quad [e^{-\alpha}(\beta' - \gamma') + 2\omega] \leq 0. \quad (16)$$

Now we can apply these requirements to our configuration at both junctions. Consider the third condition which contains the function $\beta - \gamma$ given by Eq. (6). Using it, on Σ_- with $x = x_- < 0$ we obtain

$$e^{-\alpha(u_-)} \frac{(-s' + \text{sign}\omega_0)}{s} + \frac{(1 + \Omega_- x)^2}{|x|(1 - \Omega_-^2 x^2)} \leq 0, \quad (17)$$

and on Σ_+ with $x = x_+ > 0$ we have in a similar way

$$e^{-\alpha(u_+)} \frac{(s' - \text{sign } \omega_0)}{s} + \frac{(1 + \Omega_+ x)^2}{x(1 - \Omega_+^2 x^2)} \leq 0. \quad (18)$$

Here s and $s' = ds/du$ refer to the function $s = s(k, u)$ introduced in (7).

The inequalities (17) and (18) lead to the conclusion that the NEC (hence also the WEC) cannot be satisfied on both Σ_+ and Σ_- simultaneously.

Indeed, assuming $\omega_0 > 0$, the inequality (17) can only hold if $1 - s'(k, u) < 0$ at $u = u_-$. But $s'(k, u) = \{\cosh ku, 1, \cos|k|u\}$ for $k > 0$, $k = 0$ and $k < 0$, respectively, and only at $k > 0$ we have $1 - s' < 0$. Thus at Σ_- the solution in the internal region should be taken with $k > 0$. On the contrary, (18) can hold only if $1 - s'(k, u) > 0$ at $u = u_+$, which is only possible if $k < 0$. But any particular solution describing the internal region has a fixed value of k , hence the inequalities (17) and (18) are incompatible with each other. Furthermore, if $\omega_0 < 0$, we have in the first term in (17) $-s' - 1 < 0$, making no problem; however, in (18) there is, instead, $s' + 1 > 0$, hence this inequality cannot hold whatever be the parameter k .

We conclude that the NEC is inevitably violated at least on one of the surfaces Σ_+ and Σ_- .

This general result has been obtained for any source of gravity in wormhole solutions such that $T_t^t = T_\varphi^\varphi$ in the metric (1), in particular, those given by (8). Possible solutions with other sources are certainly not ruled out. So the problem of obtaining potentially observable, stationary, nonsingular, phantom-free wormholes in GR remains open.

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